HADRON AND QUARK FORM FACTORS IN THE RELATIVISTIC HARMONIC OSCILLATOR MODEL

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Abstract

Nucleon, pion and quark form factors are studied within the relativistic harmonic oscillator model including the quark spin. It is shown that the nucleon charge, magnetic and axial form factors and the pion charge form factor can be explained with one oscillator parameter if one accounts for the scaling rule and the size of the constituent quarks.
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In this paper, we will analyze the proton electromagnetic and weak form factors (FFs) and the pion charge FF in the framework of the Relativistic Harmonic Oscillator Model (RHOM). Studies in this model were carried out in refs. [1]-[11].

In ref. [3] a method was proposed for building a covariant and gauge-invariant current within the $\tilde{U}(12) \otimes O(3,1)$ model. There remain still open problems in that approach (see, for instance, lectures [13]). In refs. [5, 6, 7] a covariant and gauge-invariant current was found for the $SU(6) \otimes O(3,1)$ model and it was shown that all the nucleon FFs can be described in this case. However, the agreement with the experimental data is here much worse than within

the nonrelativistic model that takes account of the Lorentz contraction of the nucleon wave function [14], although this latter model fails to describe the electric FF of the neutron.

The aim of the present note is to describe the nucleon FFs within RHOM using the $SU(6) \otimes O(3,1)$ scheme of derivation of the covariant and gauge-invariant currents under the assumption that the behavior of the FFs when $q^2 \to \infty$ is governed by the quark counting rules [12] and that there holds the experimentally observable scaling, i.e. $G_E^p(q^2) = G_M^p(q^2)/\mu_p = G_M^n(q^2)/\mu_n$, where $\mu_{n,p}$ are the magnetic moments of neutron and proton.

Enforcing the scaling law leads one naturally to introduce FFs for the constituent quarks. Actually, these could be regarded as interpolating functions between the static quark model and the asymptotic perturbative regime, since, as we shall see, they go to 1 both a $q^2 = 0$ and $q^2 \to \infty$.

Let us consider a system consisting of N quarks in the field of a relativistic harmonic oscillator. The corresponding wave function can be represented in the form

$$\Psi_p^{(N)}(x_1, \ldots, x_N) = \hat{A}\Phi_N(x_1, x_2, \ldots x_N)U^{(N)}(\mathbf{p}). \tag{1}$$

where $\hat{A}$ is the operator of antisymmetrization of quarks including the color degrees of freedom to be not written for simplicity, $\Phi_N(x_1, x_2, \ldots x_N)$ is a covariant space - time wave function, and $U^{(N)}(\mathbf{p})$ is a spin wave function to be described below. Let the wave function Φ_N obey the Klein-Gordon equation with a relativistic harmonic oscillator potential [1] - [11]

$$\left\{ \sum_{i=1}^{N} p_i^2 + \mathcal{K}^2 \left[\sum_{i>j}^{N} \sum_{j=1}^{N-1} (x_i - x_j)^2 \right] \right\} \Phi_N(x_1, x_2, \ldots x_N) = 0, \tag{2}$$

where $p_i = -i\partial/\partial x_i$ is a 4-momentum, $\mathcal{K}$ is the oscillator parameter, x_i is the 4-coordinate of the i-th quark (we assume all quark masses equal because of isospin invariance). Passing to the center-of-mass coordinates X and the internal variables $r_0, \ldots r_{N-1}$ and diagonalizing, one can represent (2) in the form

$$(p^2 - M_p^2)\Phi_{Nq}(r_0, r_1, \ldots r_{N-1}, p) = 0, \quad M_p^2 = -2\alpha_N a_{i\mu}^+ a_{i\mu} + \text{const}, \quad \alpha_N = \mathcal{K}N\sqrt{N}, \tag{3}$$

where $a_{i\mu}$ and $a_{i\mu}^+$ are, respectively, particle creation and annihilation operators. Under the Takabayashi condition [2], necessary for removing nonphysical oscillations along the coordinate of relative time, $p^\mu a_{i\mu}^+ \Phi_{Nq} = 0$, one gets the following solution

$$\Phi_{Nq}(r_0, r_1, \ldots r_{N-1}, p) = \left(\frac{\alpha_N}{\pi N} \right)^{N-1} \exp(\frac{\alpha_N}{2N} K^{\mu\nu} \sum_{i=1}^{N-1} r_{i\mu} r_{i\nu}), \tag{4}$$

$$\Phi_N(x_1, x_2, \ldots x_N) = \exp[ip_\mu X_\mu] \Phi_{Nq}(r_0, r_1, \ldots r_{N-1}, p), \tag{5}$$

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where p is the total momentum of the system and $K^{\mu\nu} = g^{\mu\nu} - 2p^\mu p^\nu/p^2$.

The spin wave function can be constructed in two ways (see refs. [6, 10]) by transforming the nonrelativistic spin wave function in the rest frame. The first method is to transform the wave function of every quark separately as a Dirac spinor on the basis of the Bargmann-Wigner equation. The second is to transform the wave function of the system as a whole, the minimal Pauli transformation, since in this case the wave function has a minimum number of components. The latter seems favorable, as it allows for a non-identically-zero neutron charge FF, in contrast to the first method. So, we will follow the second method according to ref. [10]. Then, the spin wave function $U^{(N)}(\mathbf{p})$ can be represented in the form

$$U^{(N)}(\mathbf{p}) \;=\; B(\mathbf{p})U^{(N)}(0), \qquad U^{(N)}(0) = \begin{pmatrix} \chi \\ 0 \end{pmatrix},$$

$$ (6) $$

$$B(\mathbf{p}) \;=\; \exp\left[\frac{b}{2|\mathbf{p}|}\rho_1(\mathbf{p}\cdot\boldsymbol{\sigma})\right] = \exp\left[\rho_1 bH\right], \qquad \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where χ is the nonrelativistic spin function of the system, $H = (\mathbf{p}\cdot\boldsymbol{\sigma})/2|\mathbf{p}|$, $b = \cosh^{-1}p_0/M$ and $\boldsymbol{\sigma} \equiv \sum_{i=1}^{N}\boldsymbol{\sigma}^i$, $\boldsymbol{\sigma}^i$ being the Pauli matrices of the i-th quark.

Based on refs.[4, 7, 10], we write the electromagnetic action in the form

$$I_{em} = \int \prod_{i=1}^{N} dx_i \sum_{k}^{N} j_{k\mu}(x_1,\ldots,x_N)A_\mu(x_k) \equiv \int dX\, J_\mu^{(N)}(X)\,A_\mu(X),$$

$$ (7) $$

where

$$j_{k\mu}(x_1,\ldots,x_N) = -i\bar{\Psi}_{p'}^{(N)}Ne_k\left[g_E(q^2)\frac{\overleftrightarrow{\partial}}{\partial x_{k\mu}} + ig_M(q^2)\sigma_{\mu\nu}^k\left(\frac{\overrightarrow{\partial}}{\partial x_{k\nu}} + \frac{\overleftarrow{\partial}}{\partial x_{k\nu}}\right)\right]\Psi_p^{(N)}.$$

$$ (8) $$

In eq. (8) $\Psi_p^{(N)}$ $(\bar{\Psi}_{p'}^{(N)})$ is the initial (final) wave function of the N quark system (1), e_k is the charge of the k-th quark, $\sigma_{\mu\nu}^k$ are the spin matrices of the k-th quark $(\sigma_{ij}^k \equiv \varepsilon_{ijl}\sigma_l^k$, $\sigma_{i4}^k = \sigma_{4i}^k \equiv \rho_1\sigma_i^k)$. Assuming the constituent quarks to be not point particles, we identify $g_E(q^2)$ and $g_M(q^2)$ with charge and magnetic quark FFs ($q = p' - p$ is the 4-momentum transferred for an N quark system with an initial (final) 4-momentum p_μ (p'_μ)). Inserting the wave function (5) into eqs. (7) and (8) and computing the integrals over the internal quark variables $r_0,\ldots,r_{N-1}$, one derives the matrix elements of the effective current for an N quark system between states of momentum p_μ (p'_μ) and spin component s (s'):

$$\left\langle p's'|J_\mu^{(N)}(0)|ps\right\rangle = \frac{I^{(N)}(q^2)}{\sqrt{2p_0p'_0}}\sum_{k=1}^{N}(\bar{U}_{s'}^{(N)}(\mathbf{p}')\Gamma_{k,\mu}U_s^{(N)}(\mathbf{p})),$$

$$ (9) $$

where

$$\Gamma_{k,\mu} = e_k[(p_\mu + p'_\mu)I_N(q^2)g_E(q^2) - iNg_M(q^2)\sigma^k_{\mu\nu}q_\nu]. \tag{10}$$

Here the overlapping integrals over space-time variables are the following

$$I^{(N)}(q^2) = \frac{1}{(1 + q^2/2M_N^2)^{(N-1)}} \exp\left[-\frac{N-1}{4\alpha_N}\left(\frac{q^2}{1 + q^2/2M_N^2}\right)\right] \tag{11}$$

$$I_N(q^2) = \frac{1 + Nq^2/2M_N^2}{1 + q^2/2M_N^2}, \tag{12}$$

M_N being the mass of the N-quark system. Using (6) one can write down eq. (9) for a three-quark system, finally obtaining expressions for the nucleon FFs, namely

$$F_M^{p,n}(q^2) \equiv \frac{G_M^p(q^2)}{\mu_p} = \frac{G_M^n(q^2)}{\mu_n} = g_M(q^2)I^{(3)}(q^2), \tag{13}$$

$$F_E^p(q^2) \equiv G_E^p(q^2) = \left[\left(1 + \frac{q^2}{4M^2}\right)I_3(q^2)g_E(q^2) - \frac{3q^2}{4M^2}g_M(q^2)\right]I^{(3)}(q^2), \tag{14}$$

$$F_E^n(q^2) \equiv G_E^n(q^2) = \frac{q^2}{2M^2}g_M(q^2)I^{(3)}(q^2). \tag{15}$$

It is also possible to compute the nucleon axial FF $F_A(q^2) = G_A(q^2)/G_A(0)$ (for details see ref. [7]),

$$F_A(q^2) = G_A(q^2)/G_A(0) = g_A(q^2)I_3(q^2)I^{(3)}(q^2), \tag{16}$$

where $g_A(q^2)$ is the constituent quark axial FF.

Note that within the $SU(6)$ model the magnetic moment of the proton equals $\mu_p = 3$, whereas for the neutron $\mu_n = -2$, somewhat different from the experimental values $\mu_p = 2.793$ and $\mu_n = -1.913$. Therefore, in the analysis of the experimental nucleon FFs we will normalize them to 1 at $q^2 = 0$. In expressions (13 - 16) $M \equiv M_3 = 0.938$ GeV is the nucleon mass.

Let us go back to the magnetic and charge nucleon FFs. According to the quark counting rules [12] the nucleon FFs decrease as q^{-4} when $q^2 \to \infty$. From inspection of formulae (11-16), it is clear that all the FFs calculated in this model have the correct asymptotic behaviour already with point-like constituent quarks. For instance, when $q^2 \to \infty$, $F_M^p(q^2) \sim g_M(q^2)/q^4$, therefore it is natural to make the minimal assumption that $g_M(q^2) = 1$. However, since from the scaling condition it also follows that $F_M^p(q^2) = F_E^p(q^2)$, we can easily derive the following expression for the quark charge FF $g_E(q^2)$:

$$g_E(q^2) = \frac{(1 + 3q^2/4M^2)(1 + q^2/2M^2)}{(1 + q^2/4M^2)(1 + 3q^2/2M^2)}g_M(q^2). \tag{17}$$

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We stress that eq. (17) is dictated by the scaling condition, which would be violated for point-like quarks, and not by the asymptotic dependence on q of the FFs, which comes out naturally from the model.

The only free parameter in this model is the oscillator parameter: fitting (13) to the experimental data for the proton FFs, we find $\alpha_3 = 0.42$ $(\text{GeV}/c)^2$, in agreement with the results of ref. [7]. From formula (14) one determines the theoretical value of the slope of the electric FF at $q^2 = 0$ to be $-dF_E^n(q^2)/dq^2|_{q^2=0} = 0.022$, which is consistent with the known experimental value 0.0202 ± 0.0003 fm^2 [15].

The standard parametrization of the nucleon FFs is in terms of a dipole function

$$F_D(q^2) = \frac{1}{\left(1 + q^2/\Lambda_{E,A}^2\right)^2}, \tag{18}$$

where for the nucleon electromagnetic FFs $\Lambda_E^2 = 0.71$ $(\text{GeV}/c)^2$ and for the axial nucleon FF $\Lambda_A = (1.032 \pm 0.036)$ GeV/c [16].

In Figs. 1 we show the comparison of our computations within RHOM with the experimental data for the nucleon electromagnetic FFs divided (except for F_E^n) by the dipole parametrization (18). Note that the proton FFs have been fitted (adjusting α_3) to experiment, whereas the others have been found without additional parameters. In the case of the neutron, the standard Galster parametrization [18] is also shown: actually, by using an effective mass of ~ 1 GeV in the RHOM calculation one can closely reproduce the Galster curve.

In Fig. 2 the dipole parametrization (solid lines) of the axial FF is compared to formula (17) with $g_A(q^2) = 1$ (dotted line) and $g_A(q^2) = g_E(q^2)$ (dashed line). Note the improved agreement in the latter case: the effect of the constituent quark size, although not dramatic, is not negligible ($\sim 10\% \div 15\%$).

It is straightforward to estimate the nucleon and quark sizes from equations (13), (14) and (17): $\bar{r}_{(p,n)} = \sqrt{<r_{(p,n)}^2>} = \sqrt{6(1/2\alpha_3 + 1/M^2)}$, $\bar{r}_q = \sqrt{<r_q^2>} = \sqrt{3/M^2}$. Thus, $\bar{r}_{(p,n)} = 0.74$ fm ($\bar{r}_{(p,n)}^{exp} \cong (0.87 \pm 0.07)$ fm), and $\bar{r}_q = 0.36$ fm. From (15), on the other hand, one finds $<r^2>_{E,n} = -3/M^2 = -0.13$ fm^2 ($<r^2>_{E,n}^{exp} = (-0.119 \pm 0.004)$ fm^2).

Concerning the axial radius, it is interesting to note that in the $g_A = 1$ case $\sqrt{<r^2>_A} = \sqrt{3/\alpha_3} = 0.53$ fm, whereas for $g_A(q^2) = g_E(q^2)$ $\sqrt{<r^2>_A} = \sqrt{3(1/\alpha_3 + 1/M^2)} = 0.64$ fm (to be compared with the experimental value of (0.65 ± 0.07) fm).

The procedure outlined above can also be applied to the quark-antiquark system. Here a source of ambiguity is due to the two possible representations for the booster in eq. (6): indeed, both $\boldsymbol{\sigma} = \boldsymbol{\sigma}^1 + \overline{\boldsymbol{\sigma}}^2$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^1 - \overline{\boldsymbol{\sigma}}^2$ lead to a valid formulation [6]. However, the latter case is known to give a simple pole-like asymptotic behaviour for the pion FF only for a special

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combination of the single-quark currents in eq. (8) (see ref. [8]). In the following we shall stick to the first formulation, where the pion charge FF reads

$$F_\pi(q^2) = g_E(q^2) I_2(q^2) I^{(2)}(q^2), \qquad (19)$$

$g_E(q^2)$ being still given by eq. (17).

Assuming the same oscillator parameter $\mathcal{K}$ for the two- and three- quark systems, the value of α_2 can be derived from α_3 since, using eq. (3), $\alpha_2/\alpha_3 = (2/3)^{(3/2)}$. Then, from $\alpha_3 = 0.42$ (GeV/c)^2, one gets $\alpha_2 = 0.23$ (GeV/c)^2. Note that in the literature both α_2 and α_3 have usually been regarded as free parameters (see, e. g., [8, 9]). A known problem is connected to the mass of the quark-antiquark system: indeed, using the physical pion mass one largely underestimates $F_\pi(q^2)$. In the literature a wide range of "$SU(6)$ symmetric masses" has been employed (0.5 GeV$\lesssim M_2 \lesssim 0.8$ GeV) [6, 7, 8, 9] and in the following we use $M_2 = 0.77$ GeV. However, it is found that this is not a critical parameter, since expression (19) is mildly dependent on M_2 for 0.4 GeV $\lesssim M_2 \lesssim 1$ GeV, giving in this range descriptions of the data of comparable quality (note that the same is not true for the case with $g_E = 1$, which is much more sensitive to M_2).

In Fig. 3 we compare with the experiment our calculations of $F_\pi(q^2)$ using either $g_E(q^2)$ given by eq. (17) and $g_E(q^2) = 1$. The improvement due to the constituent quark FF is apparent. It is again interesting to note that the pion radius is independent of M_2 and turns out to be $\bar{r}_\pi = \sqrt{<r^2>_\pi} = \sqrt{3(1/2\alpha_2 + 1/M^2)} = 0.62$ fm and $\bar{r}_\pi = \sqrt{3/2\alpha_2} = 0.50$ fm for $g_E(q^2)$ given by eq. (17) and $g_E = 1$, respectively ($\bar{r}_\pi^{exp} = (0.66 \pm 0.01)$ fm).

We summarize here our conclusions. This model gives a simple description of the experimental data on nucleon and pion FFs provided only one arbitrary parameter is used. Note that the quality of the description can be improved upon by using $g_M(q^2)$ and $g_A(q^2)$ to fit the nucleon FFs. The model could then be used, with all the parameters fixed, in investigations, for instance, of the strong πNN vertex, Δ - isobar FFs, nucleon structure functions and so on. Besides, the model can be easily extended to systems with a number of quarks larger than three, e. g. to the study of the deuteron FFs [9, 10, 11].

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Figure captions

Fig.1 The solid line is the ratio of the normalized electromagnetic FFs of the nucleon to the dipole parametrization (18); in the case of the neutron the squared electric FF is shown (solid line), compared to the Galster parametrization $F_E^n = -\mu_n(q^2/4M^2)F_D$ (dashed line). The experimental data are taken from refs. [17].

Fig.2 The normalized axial FF of the nucleon $F_A(q^2) = G_A(q^2)/G(0)$. The solid lines represent the dipole fits of the neutrino experiments and correspond to the upper and lower world average values for λ_A [16]; they are compared to the present calculation in the framework of RHOM with $g_A(q^2) = 1$ (dotted line) and $g_A(q^2) = g_E(q^2)$ (dashed line).

Fig.3 The pion charge FF (19) with $g_E(q^2)$ either given by eq. (17) (solid line) or equal to 1 (dashed line). The experimental data are taken from refs. [19].
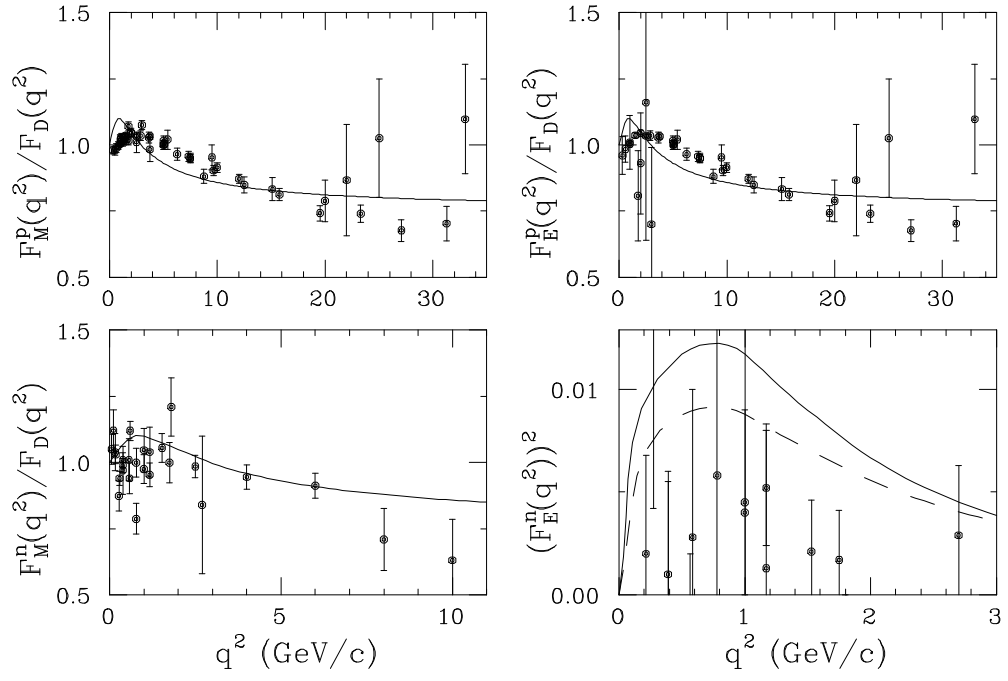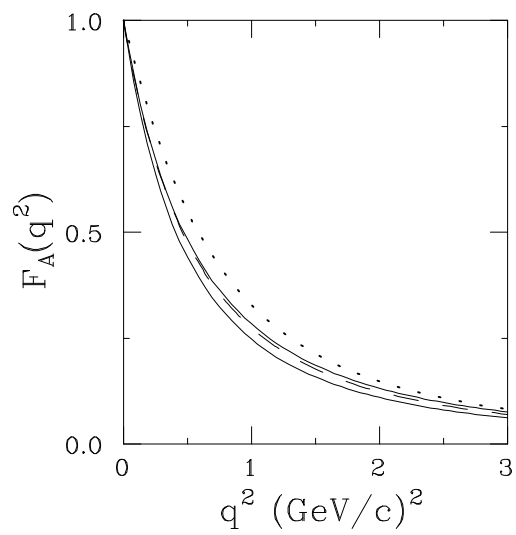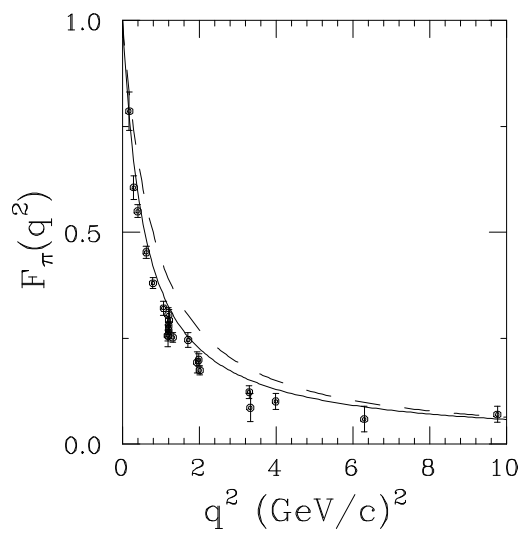
Fig. 1

Fig. 2

Fig. 3